Global monopoles and scalar fields as the electrogravity dual of Schwarzschild spacetime

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We prove that both global monopole and minimally coupled static zero mass scalar field are electrogravity dual of the Schwarzschild solution or flat space and they share the same equation of state, $T_0^0 - T_i^i = 0$. This property was however known for the global monopole spacetime while it is for the first time being established for the scalar field. In particular, it turns out that the Xanthopoulos - Zannias scalar field solution is dual to flat space.

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Phase transitions in the early universe might have given rise to several kinds of topological defects depending on the nature of the symmetry that is broken [1]. If a global SO(3) symmetry of a triplet scalar field is broken, the point like defects called global monopoles are believed to be formed. Barriola and Vilenkin [2] presented a solution which describes a global monopole at a large radial distance. It gives back the usual Schwarzschild spacetime when the monopole charge is put equal to zero.

Very recently it has been shown that in terms of the electrogravity duality, the dual to the static spherically symmetric vacuum solutions are the global monopole solutions [3]. Like the Maxwell field, the gravitational field can also be resolved into two parts, the electric part, generated by the source distribution and the magnetic part, brought into being by the motion of the sources [4]. The electric part can be further decomposed into two parts corresponding to two kinds of sources; non-gravitational energy distribution giving rise to active part and gravitational field energy to passive part. A duality transformation can be defined between them which leaves the Einstein–Hilbert action invariant if the Newtonian constant of gravitation G is replaced by $-G$ [5]. Electrogravity duality, in particular, is defined by interchange of active and passive electric parts which in familiar terms translates into interchange of the Ricci and Einstein tensors. This dulaity is at the more fundamental level the interchange between the Riemann curvature and its double dual (left and right dual). Note that contraction of Riemann is Ricci while its double dual is Einstein tensor. This is why it implies interchange of Ricci and Einstein tensors. Though the Einstein vacuum equation is invariant under this duality, it is however possible to find dual solution to the Schwarzschild solution. This happens because the effective vacuum equation can be given which is duality non-invariant and yet giving the same Schwarzschild solution [3]. The Barriola - Vilenkin global monopole solution [2] turns out to be dual to the Schwarzschild solution.

The remarkable thing is that in a similar manner it turns out that the minimally coupled static massless scalar field solution is also dual to the Schwarzschild solution. This happens perhaps because both the constructs are different manifestations of the scalar field configuration and they satisfy the same equation of state. This means that there is yet another independent way of getting to the Schwarzschild solution which leads to a different dual solution than that of the global monopole. Further both global monopole and scalar field also exist in the forms which are dual to flat space. The dual flat solutions are contained in the dual Schwarzschild solutions and could be obtained by putting the appropriate parameter to zero. Global monopole spacetime is asymptotically non-flat while the scalar field spacetime is on the other hand asymptotically flat.

Although the global monopole solution was known to be dual of the Schwarzschild solution [3], that the scalar field also shares this property is being established for the first time. It is quite remarkable and interesting. For a priori there is nothing much common between the two except being different manifestations of scalar field configuration. Both could in a sense be considered as modification of the Schwarzschild/flat space as they are in the Newtonian limit indistinguishable from the Schwarzschild/flat space. The one produces asymptotically non-flat while the other asymptotically flat modification.

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Let us begin by defining the three kinds of energy density as follows: (i) Energy density as measured by a static observer, $\rho = T_{ab}u^a u^b; u_a u^a = 1$. (ii) Null energy density, $\rho_n = T_{ab}k^a k^b; k_a k^a = 0$. (iii) Timelike convergence density, $\rho_t = (T_{ab} - 1/2\ T g_{ab})u^a u^b$. We shall use them to characterize effective vacuum and its dual space. It turns out that vacuum for spherically symmetric spacetime could effectively be characterized by the condition, $\rho = 0, \rho_n = 0$. This definition is physically illuminating as it is parallel to the Newtonian definition which simply demands vanishing of matter density [6]. Since in GR, we want both timelike and null particles to interact gravitationally, hence density relative to both of them must vanish in vacuum. $\rho = 0$ would imply $G^0_0 = 0$ while $\rho_n = 0$ would imply either $G^0_0 = G^1_1$ for radial or $G^0_0 = G^2_2$ for circular or $G^0_0 = G^1_1 = G^2_2$ for non radial photons. Of course the last alternative with $\rho = 0$ would imply the usual vacuum equation $R_{ab} = 0$. However either of the other two with vanishing ρ is good enough to give the Schwarzschild solution. We thus have the effective empty space equation as

$$G^0_0 = G^1_1 = 0, \tag{1}$$

or alternatively

$$G^0_0 = G^2_2 = 0. \tag{2}$$

Note that for the spherical symmetry, the vacuum equation implies only two independent equations, which would not in contrast to $R_{ab} = 0$ be in general electrogravity duality ($R_{ab} \leftrightarrow G_{ab}$) invariant. Thus under the duality transformation, the above equations would give rise to the two sets of the dual equations which would admit two distinct dual solutions.

Of course both of the above equations admit the well-known Schwarzschild solution as the general solution, which is given by the metric,

$$ds^2 = Adt^2 - A^{-1}dr^2 - r^2 d\omega^2,\ d\omega^2 = d\theta^2 + sin^2\theta d\varphi^2, \tag{3}$$

where

$$A = 1 - 2m/r, \tag{4}$$

with m denoting the mass of the Schwarzschild particle.

The electrogravity duality is defined [3] by the interchange of active and passive electric parts which translates in the familiar terms to the interchange of Ricci and Einstein tensors, viz.

$$R_{ab} \leftrightarrow G_{ab}. \tag{5}$$

The equation dual to the effective vacuum equations would be

$$R^0_0 = R^1_1 = 0, \tag{6}$$

corresponding to eqn(1) and

$$R^0_0 = R^2_2 = 0. \tag{7}$$

corresponding to eqn(2). For densities, the duality transformation implies $\rho \leftrightarrow \rho_t, \rho_n \rightarrow \rho_n$.

The former set (6) admits the general solution as the Barriola - Vilenkin global monopole solution [2] given by

$$ds^2 = (A - \eta^2)dt^2 - (A - \eta^2)^{-1}dr^2 - r^2 d\omega^2, \tag{8}$$

where η marks the scale of symmetry breaking. The stresses are given by

$$T^0_0 = T^1_1 = \eta^2/r^2, \tag{9}$$

which is the form required at large distance by global monopole [2] . Note that it is not flat even when $m = 0$ and hence is asymptotically non-flat. In this limit it turns out to be dual to flat space.

The latter set (7) would admit the general solution given by

$$ds^2 = A^n dt^2 - A^{-n} dr^2 - r^2 A^{1-n} d\omega^2, \tag{10}$$

where A is as given in eqn (4). Interestingly this is also dual to the Schwarzschild solution to which it would reduce when $n = \pm 1$. It reduces to flat space when $m = 0$ and hence is clearly asymptotically flat.

The stress tensor for a scalar field reads as

$$T_{ab} = \phi_{,a}\phi_{,b} - \frac{1}{2}\phi_{,c}\phi^{,c} g_{ab}, \tag{11}$$

which for the static spherically symmetric field would have $T_0^0 = -T_1^1 = T_2^2$. For the above metric, we shall thus have

$$T_0^0 = -T_1^1 = T_2^2 = (1-n^2)\frac{m^2}{r^4 A^{(2-n)}}, \tag{12}$$

and the scalar field itself is given by

$$\phi = (1 - 2m/r)^{\sqrt{\frac{1-n^2}{2}}}. \tag{13}$$

Clearly when $n = 1$, it reduces to the Schwarzschild solution while for $n = -1$ it can be brought to the standard Schwarzschild form by letting $m \to -m$. Note that for acceleration to be attractive for the metric (10, m must be positive while the scalar field energy density in eqn (12) is insensitive to its sign. The above solution thus reduces to the vacuum spacetime which is respectively the Schwarzschild (m)and anti-Schwarzschild ($-m$) according to $n = 1$ and $n = -1$.

On the other hand the case $n = 0$ would be the case of dual flat (dual set (7) for the gauge $g_{00} = 1$) scalar field solution. It could be cast into the form,

$$ds^2 = dt^2 - dr^2 - (r^2 - m^2)d\omega^2. \tag{14}$$

and the stress tensor would reduce to

$$G_0^0 = -G_1^1 = G_2^2 = -\frac{m^2}{(r^2 - m^2)^2}, \tag{15}$$

with the scalar field,

$$\phi = \frac{1}{\sqrt{2}} ln \frac{r-m}{r+m}. \tag{16}$$

It is the Xanthopoulos and Zannias (XZ) scalar field solution [7] which is dual to flat space.

We have thus two independent two parameter families of solutions dual to the Schwarzschild solution which contain the dual flat families. One of them is asymptotically flat while the other is not. Both share the same equation of state, $\rho_t = 0$.

To the scalar field a global monopole could readily be added by a general prescription due to Dadhich and Patel [8] for any spherically symmetric solution. We just need to multiply the angular part of the metric by a constant. Thus in the above XZ scalar field metric (14) a global monopole could be added simply by writing $k^2(r^2 - m^2)$ in place of $r^2 - m^2$. It would have the superposition of the two stress tensors; $T_{ab} = T_{ab}(GM) + T_{ab}(SF)$. For the global monopole we have,

$$T_0^0 = T_1^1 = \frac{\left(1 - 1/k^2\right)}{r^2 - m^2}. \tag{17}$$

and for the XZ scalar field

$$T_0^0 = -T_1^1 = T_2^2 = \frac{m^2}{(r^2 - m^2)^2}. \tag{18}$$

It thus represents a massless global monopole superposed onto the XZ scalar field [9]. It reduces to the massless global monopole (8) for $m = 0$ and to the XZ scalar field (14) for $k = 1$.

Further, it is similarly also possible to superpose the other two dual solutions of global monopole (8) and of the scalar field (10). The resulting metric would read as

$$ds^2 = (A - \eta^2)^n - (A - \eta^2)^{-n} dr^2 - r^2 (A - \eta^2)^{1-n} d\omega^2, \tag{19}$$

which is identified to be the global monopole with a scalar field as given by Banerjee et al [10]. Note that the above metric could be transformed to the form in which the angular part of the scalar field metric is multiplied by a constant. The scalar field ϕ as given by (13) and the equation of state $\rho_t = 0$ would remain unaltered under this superposition. The stress tensor for the metric (19) would be given by

$$T_0^0 = \frac{\eta^2/r^2}{(A - \eta^2)^{1-n}} + T_2^2 \tag{20}$$

$$T_1^1 = T_0^0 - 2T_2^2 \tag{21}$$

$$T_2^2 = (1 - n^2) \frac{m^2/r^4}{(A - \eta^2)^{(2-n)}} \tag{22}$$

which is the superposition of the global monopole and the scalar field. Clearly it is the global monopole for $m = 0$ and the scalar field for $\eta = 0$.

By utilising the fact that the Schwarzschild solution could be obtained by two distinct sets of equations which are electrogravity duality non-invariant, we have shown that both the global monopole and the scalar field solutions are dual to the Schwazschild solution. They also include solutions dual to flat space. The other interesting feature is that the global monopole spacetimes are asymptotically non-flat while those of the scalar field are asymptotically flat. It is also remarkable that the global monopole and the scalar field could be superposed in which the resulting stress tensor is the sum of the two, and the only change in the scalar field metric is a constant multiplying the angular part. This is in accordance with the general prescription of Dadhich and Patel [8] for intrduction of a global monopole like stresses in any spherically symmetric spacetime. All the spacetimes discussed here satisfy the same equation of state $\rho_t = 0$, including the one in which the two fields are superposed. Under the superposition the solution for the scalar field remains undisturbed.

From the stresses in eqns (12) and (15), it is clear that the scalar field has energy density which falls of as $1/r^4$, like the gravitational field energy density, which could be made both positive or negative by choosing the sign of constant m^2. It is however generally taken to be positive as is the case for the XZ solution. Thus the scalar field solution represents a Schwarzschild particle sitting in a positive energy distribution. Very recently Dadhich [11] has shown that for a distribution engulfing an isolated object, the "positive" energy condition is that its energy is negative. Its effect like that of gravitational field energy would be felt only through the space curvature. Since gravitational field energy is negative , it produces negative curvature in space which works in unison with the attraction produced by the gradient of potential. Thus only when energy density is negative, it would act in conformity with the acceleration produced by the gradient of potential. Since the scalar field energy density is positive it would produce positive space curvature which would tend to oppose the effect of the potential. This is similar to the case of charged black hole where positive electric field energy density produces the repulsive effect counteracting the effect of mass. A similar situation also obtains for a black hole on the brane in the currently popular Randall-Sundrum brane world model [12]. The back reaction of the bulk on the brane effectively produces on the brane a trace free matter field through the Weyl curvature of the bulk. That is the black hole on the brane has the similar enviornmemnt [13] of energy density falling off as $1/r^4$. Here also if we let energy density of scalar field to be negative, it would then contribute positively to the field of the Schwazschild particle as in the brane world model. That is we can approximately mimick the black hole on the brane simply by a negative energy density scalar field. This is an interesting and novel way of looking at the scalar field solution. We hope to work further on this track in future.

The XZ scalar field solution (14) could be cast in the following interesting form,

$$ds^2 = dt^2 - (1 + m^2/r^2)^{-1} dr^2 - r^2 d\omega^2, \tag{23}$$

which clearly shows its asymptotic flatness. There is no central mass to produce the radial acceleration. Hence the motion for both the dual flat global monopole (metric (8) with $m = 0$) as well as the dual flat XZ scalar field solutions

would be solely guided by the space curvature. So long as the energy density is positive, there would occur repulsive effect [11] as is well-known for the global monopole [14]. The contrary would be the case for negative energy density.

Finally it is worth remembering that the duality character of the scalar field solutions unlike that of global monopole is being establishsed for the first time. It would be interesting to seek scalar field dual solutions in all those cases where global monopole solutions exist, for instance in the NUT space [15], Kaluza - Klein space [16] and 2+1 gravity [17].

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